Covariant Derivatives of Extensor Fields

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Abstract

A simple theory of the covariant derivatives, deformed derivatives and relative covariant derivatives of *extensor* fields is present using algebraic and analytical tools developed in previous papers. Several important formulas are derived.

1 Introduction

A simple theory of the covariant derivatives, deformed derivatives and relative covariant derivatives for $\bigwedge TM$ and $\bigwedge T^*M$ valued *extensor* fields on a arbitrary smooth manifold M, which are important in, e.g., intrinsic formulations of geometrical theories of the gravitational field and in the Lagrangian formalism of field theories [4] is presented using algebraic and analytical tools developed in previous papers [1, 2, 3]. Useful properties of these concepts are present with details.

2 Extensor Fields

Let U be an open set on the finite dimensional smooth manifold M (i.e., $\dim M = n$ with $n \in \mathbb{N}$). As usual, the ring (with identity) of the smooth scalar fields on U will denoted by $\mathcal{S}(U)$. The module over $\mathcal{S}(U)$ of the smooth vector fields on U will be symbolized as $\mathcal{V}(U)$.

The modules over $\mathcal{S}(U)$ of the smooth[1] multivector fields on U and the smooth multiform fields on U will be respectively denoted by $\bigwedge \mathcal{V}(U)$ and $\bigwedge \mathcal{V}^*(U)$.

A multivector extensor mapping

$$\tau : U \longrightarrow \bigcup_{p \in U} ext_{\,k}^{\,l}(T_pM) \tag{1}$$

such that for each $p \in U$, $\tau_{(p)} \in ext_{\,k}^{\,l}(T_pM)$ is called a *multivector extensor field of k multivector and l multiform variables on U.*

A multiform extensor mapping

$$\upsilon : U \longrightarrow \bigcup_{p \in U} \overset{*}{ext}{}_{\,k}^{\,l}(T_pM) \tag{2}$$

such that for each $p \in U$, $\upsilon_{(p)} \in \overset{*}{ext}{}_{\,k}^{\,l}(T_pM)$ is called a *multiform extensor field of k multivector and l multiform variables on U.*

[1]In this paper, smooth menas $\mathcal{C}^\infty$-differentiable, or at least enough differentiable for our statements to hold.

In the above formulas, $\overset{l}{ext}_{k}(T_pM)$ is a short notation for the space of multivector extensors of k multivector and l multiform variables over T_pM, i.e., for each $p \in U$:

$$\overset{l}{ext}_{k}(T_pM) := ext(\bigwedge_1^{\diamond} T_pM, \ldots, \bigwedge_k^{\diamond} T_pM, \bigwedge_1^{\diamond} T_p^{\star}M, \ldots, \bigwedge_l^{\diamond} T_p^{\star}M; \bigwedge^{\diamond} T_pM),$$

and $\overset{*}{ext}{}_{k}^{l}(T_pM)$ is a shorth notation for the space of multiform *extensors* of k multivector and l multiform variables over T_pM, i.e., for each $p \in U$:

$$\overset{*}{ext}{}_{k}^{l}(T_pM) := ext(\bigwedge_1^{\diamond} T_pM, \ldots, \bigwedge_k^{\diamond} T_pM, \bigwedge_1^{\diamond} T_p^{\star}M, \ldots, \bigwedge_l^{\diamond} T_p^{\star}M; \bigwedge^{\diamond} T_p^{\star}M).$$

Let us denote the smooth multivector fields: $U \ni p \longmapsto X_{1(p)} \in \bigwedge_1^{\diamond} T_pM, \ldots$, $U \ni p \longmapsto X_{k(p)} \in \bigwedge_k^{\diamond} T_pM$, and $U \ni p \longmapsto X_{(p)} \in \bigwedge^{\diamond} T_pM$, respectively by $\bigwedge_1^{\diamond} \mathcal{V}(U), \ldots$, $\bigwedge_k^{\diamond} \mathcal{V}(U)$, and $\bigwedge^{\diamond} \mathcal{V}(U)$. Let us denote the smooth multiform fields: $U \ni p \longmapsto \Phi^1_{(p)} \in \bigwedge_1^{\diamond} T_p^{*}M, \ldots$, $U \ni p \longmapsto \Phi^l_{(p)} \in \bigwedge_l^{\diamond} T_p^{*}M$, and $U \ni p \longmapsto \Phi^l_{(p)} \in \bigwedge^{\diamond} T_p^{*}M$, by $\bigwedge_1^{\diamond} \mathcal{V}^{*}(U)$, $\ldots$, $\bigwedge_l^{\diamond} \mathcal{V}^{*}(U)$ and $\bigwedge^{\diamond} \mathcal{V}^{*}(U)$.

Such a multivector extensor field τ will be said to be smooth, if and only if, for all $X_1 \in \bigwedge_1^{\diamond} \mathcal{V}(U), \ldots, X_k \in \bigwedge_k^{\diamond} \mathcal{V}(U)$, and for all $\Phi^1 \in \bigwedge_1^{\diamond} \mathcal{V}^{*}(U), \ldots, \Phi^l \in \bigwedge_l^{\diamond} \mathcal{V}^{\star}(U)$, the multivector mapping defined by

$$U \ni p \longmapsto \tau_{(p)}(X_{1(p)}, \ldots, X_{k(p)}, \Phi^1_{(p)}, \ldots, \Phi^l_{(p)}) \in \bigwedge^{\diamond} T_pM \tag{3}$$

is a smooth multivector field on U, (i.e., an object living on $\bigwedge^{\diamond} \mathcal{V}(U)$).

Such a multiform extensor field υ will be said to be smooth , if and only if, for all $X_1 \in \bigwedge_1^{\diamond} \mathcal{V}(U), \ldots, X_k \in \bigwedge_k^{\diamond} \mathcal{V}(U)$, and for all $\Phi^1 \in \bigwedge_1^{\diamond} \mathcal{V}^{*}(U), \ldots, \Phi^l \in \bigwedge_l^{\diamond} \mathcal{V}^{*}(U)$, the multiform mapping defined by

$$U \ni p \longmapsto \upsilon_{(p)}(X_{1(p)}, \ldots, X_{k(p)}, \Phi^1_{(p)}, \ldots, \Phi^l_{(p)}) \in \bigwedge^{\diamond} T_p^{\star}M \tag{4}$$

is a smooth multiform field on U, (i.e., an object living on $\overset{\diamond}{\bigwedge} \mathcal{V}^{\star}(U)$).

We emphasize[2] that according with the definitions of smoothness as given above, a smooth $\binom{l}{k}$ multivector extensor field on U *can be identified* to a $\binom{l}{k}$ multivector extensor over $\mathcal{V}(U)$. It is also true that a smooth $\binom{l}{k}$ multiform extensor field on U *can be properly seen* as a $\binom{l}{k}$ multiform extensor over $\mathcal{V}(U)$.

Thus, the set of smooth $\binom{l}{k}$ multivector extensor fields on U is just a module over $\mathcal{S}(U)$ which could be denoted by $ext{}_k^l\mathcal{V}(U)$. And, the set of smooth $\binom{l}{k}$ multiform extensor fields on U is also a module over $\mathcal{S}(U)$ which can be symbolized as $\overset{*}{ext}{}_k^l\mathcal{V}(U)$.

2.1 Algebras of Extensor Fields

We define now the exterior products of smooth multivector extensor fields on U and smooth multiform extensor fields on U. We also present the definitions of smooth multivector extensor fields on U with smooth multiform extensor fields on U.

The exterior product of either multivector extensor fields or multiform extensor fields τ and σ is defined as

$$(\tau \wedge \sigma)_{(p)} = \tau_{(p)} \wedge \sigma_{(p)} \tag{5}$$

for every $p \in U$.

Each module over $\mathcal{S}(U)$ of either the smooth multivector extensor fields on U or the smooth multiform extensor fields on U endowed with the respective exterior product is an associative algebra.

The duality scalar product of a multiform extensor field τ with a multivector extensor field σ is a scalar extensor field $\langle \tau, \sigma \rangle$ defined by

$$\langle \tau, \sigma \rangle_{(p)} = \left\langle \tau_{(p)}, \sigma_{(p)} \right\rangle, \tag{6}$$

for every $p \in U$.

The duality left contracted product of a multiform extensor field τ with a multivector extensor field σ (or, a multivector extensor field τ with a multiform extensor field σ) is the multivector extensor field (respectively, the multiform extensor field) denoted by $\langle \tau, \sigma |$ and defined by

$$\langle \tau, \sigma |_{(p)} = \left\langle \tau_{(p)}, \sigma_{(p)} \right|, \tag{7}$$

[2] A *short name* for a multivector (or, multiform) extensor of k multivector and l multiform variables could be: a $\binom{l}{k}$ multivector (respectively, multiform) extensor.

for every $p \in U$.

The duality right contracted product of a multiform extensor field τ with a multivector extensor field σ (or, a multivector extensor field τ with a multiform extensor field σ) is the multiform extensor field (respectively, the multivector extensor field) named as $|\tau, \sigma\rangle$, and defined by

$$|\tau, \sigma\rangle_{(p)} = \left|\tau_{(p)}, \sigma_{(p)}\right\rangle \tag{8}$$

for every $p \in U$.

Each duality contracted product of smooth multivector extensor fields on U with smooth multiform fields on U yields a non-associative algebra.

3 Covariant Derivative of Extensor Fields

Let $\langle U, \Gamma\rangle$ be a parallelism structure [2] on U, and let us take $a \in \mathcal{V}(U)$. The *a-Directional Covariant Derivatives* (*a-DCD*), associated with $\langle U, \Gamma\rangle$, of a smooth *multivector* extensor field on U or a smooth *multiform* extensor field on U are the mappings

$$ext_{\,k}^{\,l}\,\mathcal{V}(U) \ni \tau \longmapsto \nabla_a \tau \in ext_{\,k}^{\,l}\,\mathcal{V}(U),$$

and

$$\overset{*}{ext}{}_{\,k}^{\,l}\,\mathcal{V}(U) \ni \tau \longmapsto \nabla_a \tau \in \overset{*}{ext}{}_{\,k}^{\,l}\,\mathcal{V}(U),$$

such that for all $X_1 \in \bigwedge_1^{\diamond}\mathcal{V}(U), \ldots, X_k \in \bigwedge_k^{\diamond}\mathcal{V}(U)$, and for all $\Phi^1 \in \bigwedge_1^{\diamond}\mathcal{V}^*(U), \ldots, \Phi^l \in \bigwedge_l^{\diamond}\mathcal{V}^*(U)$ we have

$$\begin{aligned}
\nabla_a \tau(X_1, ..., X_k, \Phi^1, ..., \Phi^l) = {} & \nabla_a(\tau(X_1, ..., X_k, \Phi^1, ..., \Phi^l)) \\
& - \tau(\nabla_a X_1, ..., X_k, \Phi^1, ..., \Phi^l) - \cdots \\
& - \tau(X_1, ..., \nabla_a X_k, \Phi^1, ..., \Phi^l) \\
& - \tau(X_1, ..., X_k, \nabla_a \Phi^1, ..., \Phi^l) - \cdots \\
& - \tau(X_1, ..., X_k, \Phi^1, ..., \nabla_a \Phi^l).
\end{aligned} \tag{9}$$

The covariant derivative of smooth multivector (or multiform) extensor fields has two basic properties.

- For $f \in \mathcal{S}(U)$, and $a, b \in \mathcal{V}(U)$, and $\tau \in ext_{k}^{l}\mathcal{V}(U)$ (or $\tau \in \overset{*}{ext}_{k}^{l}\mathcal{V}(U)$)

$$\nabla_{a+b}\tau = \nabla_a\tau + \nabla_b\tau \tag{10}$$

$$\nabla_{fa}\tau = f\nabla_a\tau. \tag{11}$$

- For $f \in \mathcal{S}(U)$, and $a \in \mathcal{V}(U)$, and $\tau, \sigma \in ext_{k}^{l}\mathcal{V}(U)$ (or $\tau, \sigma \in \overset{*}{ext}_{k}^{l}\mathcal{V}(U)$)

$$\nabla_a(\tau + \sigma) = \nabla_a\tau + \nabla_a\sigma, \tag{12}$$

$$\nabla_a(f\tau) = (af)\tau + f\nabla_a\tau. \tag{13}$$

The covariant differentiation of the exterior product of smooth multivector (or multiform) extensor fields satisfies the Leibniz's rule.

- For all $\tau \in ext_{k}^{l}\mathcal{V}(U)$ and $\sigma \in ext_{r}^{s}\mathcal{V}(U)$ (or, $\tau \in \overset{*}{ext}_{k}^{l}\mathcal{V}(U)$ and $\sigma \in \overset{*}{ext}_{r}^{s}\mathcal{V}(U)$), it holds

$$\nabla_a(\tau \wedge \sigma) = (\nabla_a\tau) \wedge \sigma + \tau \wedge (\nabla_a\sigma). \tag{14}$$

Proof. Without loss of generality, we prove this statement only for multivector extensor fields $(X, \Phi) \longmapsto \tau(X, \Phi)$ and $(Y, \Psi) \longmapsto \sigma(Y, \Psi)$. Using Eq.(9), we can write

$$\begin{aligned}
&\nabla_a(\tau \wedge \sigma)(X, Y, \Phi, \Psi) \\
&= \nabla_a((\tau \wedge \sigma)(X, Y, \Phi, \Psi)) \\
&- (\tau \wedge \sigma)(\nabla_a X, Y, \Phi, \Psi) - (\tau \wedge \sigma)(X, \nabla_a Y, \Phi, \Psi) \\
&- (\tau \wedge \sigma)(X, Y, \nabla_a\Phi, \Psi) - (\tau \wedge \sigma)(X, Y, \Phi, \nabla_a\Psi).
\end{aligned}$$

Using Eq.(5) and recalling Leibniz's rule for the covariant differentiation of the exterior product of multivector fields, we have

$$\begin{aligned}
&\nabla_a(\tau \wedge \sigma)(X, Y, \Phi, \Psi) \\
&= \nabla_a(\tau(X, \Phi)) \wedge \sigma(Y, \Psi) + \tau(X, \Phi) \wedge \nabla_a(\sigma(Y, \Psi)) \\
&- \tau(\nabla_a X, \Phi) \wedge \sigma(Y, \Psi) - \tau(X, \Phi) \wedge \sigma(\nabla_a Y, \Psi) \\
&- \tau(X, \nabla_a\Phi) \wedge \sigma(Y, \Psi) - \tau(X, \Phi) \wedge \sigma(Y, \nabla_a\Psi),
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\nabla_a(\tau\wedge\sigma)(X,Y,\Phi,\Psi)\\
&=(\nabla_a(\tau(X,\Phi))-\tau(\nabla_a X,\Phi)-\tau(X,\nabla_a\Phi))\wedge\sigma(Y,\Psi)\\
&+\tau(X,\Phi)\wedge(\nabla_a(\sigma(Y,\Psi))-\sigma(\nabla_a Y,\Psi)-\sigma(Y,\nabla_a\Psi)).
\end{aligned}$$

Then, using once again Eq.(9) and Eq.(5), the expected result follows. ■

The covariant differentiation of the duality scalar product of smooth extensor fields satisfies the Leibniz's rule.

- For all $\tau\in\overset{*}{ext}{}_k^{\,l}\mathcal{V}(U)$ and $\sigma\in ext_r^{\,s}\mathcal{V}(U)$ (or, $\tau\in ext_k^{\,l}\mathcal{V}(U)$ and $\tau\in\overset{*}{ext}{}_r^{\,s}\mathcal{V}(U)$), we have that

$$\nabla_a\left\langle\tau,\sigma\right\rangle=\left\langle\nabla_a\tau,\sigma\right\rangle+\left\langle\tau,\nabla_a\sigma\right\rangle. \tag{15}$$

Proof. Without loss of generality, we prove this statement only for a multiform extensor field τ and a multivector extensor field σ such that $(X,\Phi)\longmapsto\tau(X,\Phi)$ and $(Y,\Psi)\longmapsto\sigma(Y,\Psi)$. Using Eq.(**9**), we can write

$$\begin{aligned}
&\nabla_a\left\langle\tau,\sigma\right\rangle(X,Y,\Phi,\Psi)\\
&=\nabla_a(\left\langle\tau,\sigma\right\rangle(X,Y,\Phi,\Psi))\\
&-\left\langle\tau,\sigma\right\rangle(\nabla_a X,Y,\Phi,\Psi)-\left\langle\tau,\sigma\right\rangle(X,\nabla_a Y,\Phi,\Psi)\\
&-\left\langle\tau,\sigma\right\rangle(X,Y,\nabla_a\Phi,\Psi)-\left\langle\tau,\sigma\right\rangle(X,Y,\Phi,\nabla_a\Psi).
\end{aligned}$$

Next, using Eq.(**6**) and recalling Leibniz's rule for the covariant differentiation of the duality scalar product of multiform fields with multivector fields, we have

$$\begin{aligned}
&\nabla_a\left\langle\tau,\sigma\right\rangle(X,Y,\Phi,\Psi)\\
&=\left\langle\nabla_a(\tau(X,\Phi)),\sigma(Y,\Psi)\right\rangle+\left\langle\tau(X,\Phi),\nabla_a(\sigma(Y,\Psi))\right\rangle\\
&-\left\langle\tau(\nabla_a X,\Phi),\sigma(Y,\Psi)\right\rangle-\left\langle\tau(X,\Phi),\sigma(\nabla_a Y,\Psi)\right\rangle\\
&-\left\langle\tau(X,\nabla_a\Phi),\sigma(Y,\Psi)\right\rangle-\left\langle\tau(X,\Phi),\sigma(Y,\nabla_a\Psi)\right\rangle,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\nabla_a\left\langle\tau,\sigma\right\rangle(X,Y,\Phi,\Psi)\\
&=\left\langle(\nabla_a(\tau(X,\Phi))-\tau(\nabla_a X,\Phi)-\tau(X,\nabla_a\Phi)),\sigma(Y,\Psi)\right\rangle\\
&+\left\langle\tau(X,\Phi),(\nabla_a(\sigma(Y,\Psi))-\sigma(\nabla_a Y,\Psi)-\sigma(Y,\nabla_a\Psi))\right\rangle.
\end{aligned}$$

Then, using once again Eq.(9) and Eq.(6), the expected result follows. ■

The covariant differentiation of each one of the duality contracted products of smooth extensor fields satisfies the Leibniz's rule.

- For all $\tau \in \overset{*}{ext}{}_k^l \mathcal{V}(U)$ and $\sigma \in ext_r^s \mathcal{V}(U)$ (or, $\tau \in ext_k^l \mathcal{V}(U)$ and $\tau \in \overset{*}{ext}{}_r^s \mathcal{V}(U)$), it holds

$$\nabla_a \left\langle \tau, \sigma \right| = \left\langle \nabla_a \tau, \sigma \right| + \left\langle \tau, \nabla_a \sigma \right|, \tag{16}$$

$$\nabla_a \left\langle \sigma, \tau \right| = \left\langle \nabla_a \sigma, \tau \right| + \left\langle \sigma, \nabla_a \tau \right|. \tag{17}$$

- For all $\tau \in \overset{*}{ext}{}_k^l \mathcal{V}(U)$ and $\sigma \in ext_r^s \mathcal{V}(U)$ (or, $\tau \in ext_k^l \mathcal{V}(U)$ and $\tau \in \overset{*}{ext}{}_r^s \mathcal{V}(U)$), it holds

$$\nabla_a \left| \tau, \sigma \right\rangle = \left| \nabla_a \tau, \sigma \right\rangle + \left| \tau, \nabla_a \sigma \right\rangle, \tag{18}$$

$$\nabla_a \left| \tau, \sigma \right\rangle = \left| \nabla_a \tau, \sigma \right\rangle + \left| \tau, \nabla_a \sigma \right\rangle. \tag{19}$$

Proof. We present only the proof of the property given by Eq.(16). Without loss of generality, we prove this statement only for a multiform extensor field τ and a multivector extensor field σ such that $(X, \Phi) \longmapsto \tau(X, \Phi)$ and $(Y, \Psi) \longmapsto \sigma(Y, \Psi)$.

$$\begin{aligned}
\nabla_a \left\langle \tau, \sigma \right|(X, Y, \Phi, \Psi) &= \nabla_a \left(\left\langle \tau, \sigma \right|(X, Y, \Phi, \Psi) \right) - \left\langle \tau, \sigma \right|(\nabla_a X, Y, \Phi, \Psi) \\
&\quad - \left\langle \tau, \sigma \right|(X, \nabla_a Y, \Phi, \Psi) - \left\langle \tau, \sigma \right|(X, Y, \nabla_a \Phi, \Psi) \\
&\quad - \left\langle \tau, \sigma \right|(X, Y, \Phi, \nabla_a \Psi)
\end{aligned}$$

or recalling that $\left\langle \tau, \sigma \right|_{(p)} = \left\langle \tau_{(p)}, \sigma_{(p)} \right|.$,

$$\begin{aligned}
\nabla_a \left\langle \tau, \sigma \right|(X, Y, \Phi, \Psi) &= \nabla_a \left(\left\langle \tau(X, \Phi), \sigma(Y, \Psi) \right| \right) - \left\langle \tau(\nabla_a X, \Phi), \sigma(Y, \Psi) \right| \\
&\quad - \left\langle \tau(X, \Phi), \sigma(\nabla_a Y, \Psi) \right| - \left\langle \tau(X, \nabla_a \Phi), \sigma(Y, \Psi) \right| \\
&\quad - \left\langle \tau(X, \Phi), \sigma(Y, \nabla_a \Psi) \right| .
\end{aligned} \tag{20}$$

On the other hand, from [2], we can write

$$\nabla_a \left(\left\langle \tau(X, \Phi), \sigma(Y, \Psi) \right| \right) = \left\langle \nabla_a \tau(X, \Phi), \sigma(Y, \Psi) \right| + \left\langle \tau(X, \Phi), \nabla_a \sigma(Y, \Psi) \right| ,$$

Eq.(20) can by written as

$$\begin{aligned}
&\nabla_a \left\langle \tau, \sigma \right|(X, Y, \Phi, \Psi) \\
&= \left\langle \nabla_a \tau(X, \Phi) - \tau(\nabla_a X, \Phi) - \tau(X, \nabla_a \Phi), \sigma(Y, \Psi) \right| \\
&+ \left\langle \tau(X, \Phi), \nabla_a \sigma(Y, \Psi) - \sigma(\nabla_a Y, \Psi) - \sigma(Y, \nabla_a \Psi) \right| \\
&= \left\langle (\nabla_a \tau)(X, \Phi), \sigma(Y, \Psi) \right| + \left\langle \tau(X, \Phi), (\nabla_a \sigma)(Y, \Psi) \right| \\
&= \left\langle \nabla_a \tau, \sigma \right|(X, Y, \Phi, \Psi) + \left\langle \tau, \nabla_a \sigma \right|(X, Y, \Phi, \Psi) ,
\end{aligned}$$

which proves our result. ■

Finally we prove that the duality adjoint operator commutes with the *a-DCDO*, i.e.:

- If τ is any one of the four smooth one-variable extensor fields on U, then
$$(\nabla_a \tau)^{\triangle} = \nabla_a \tau^{\triangle}. \tag{21}$$

Proof. Without loss of generality, we prove this statement only for $\tau \in ext(\bigwedge_1^{\diamond} \mathcal{V}(U), \bigwedge^{\diamond} \mathcal{V}(U))$.

Let us $X \in \bigwedge_1^{\diamond} \mathcal{V}(U)$ and $\Phi \in \bigwedge^{\diamond} \mathcal{V}^*(U)$. We must prove that

$$\left\langle \nabla_a \tau^{\triangle}(\Phi), X \right\rangle = \left\langle \Phi, \nabla_a \tau(X) \right\rangle.$$

By using Eq.(9) and recalling the Leibniz's rule for the covariant differentiation of the duality scalar product of multiform fields with multivector fields, we can write

$$\left\langle \nabla_a \tau^{\triangle}(\Phi), X \right\rangle = \left\langle \nabla_a (\tau^{\triangle}(\Phi)), X \right\rangle - \left\langle \tau^{\triangle}(\nabla_a(\Phi)), X \right\rangle,$$

i.e.,

$$\left\langle \nabla_a \tau^{\triangle}(\Phi), X \right\rangle = a \left\langle \tau^{\triangle}(\Phi), X \right\rangle - \left\langle \tau^{\triangle}(\Phi), \nabla_a X \right\rangle - \left\langle \tau^{\triangle}(\nabla_a \Phi), X \right\rangle.$$

Recalling now the fundamental property of the duality *adjoint*, we get

$$\left\langle \nabla_a \tau^{\triangle}(\Phi), X \right\rangle = a \left\langle \Phi, \tau(X) \right\rangle - \left\langle \Phi, \tau(\nabla_a X) \right\rangle - \left\langle \nabla_a \Phi, \tau(X) \right\rangle.$$

Using once again the Leibniz's rule for the covariant differentiation of the duality scalar product we get

$$\left\langle \nabla_a \tau^{\triangle}(\Phi), X \right\rangle = \left\langle \Phi, \nabla_a(\tau(X)) \right\rangle - \left\langle \Phi, \tau(\nabla_a X) \right\rangle,$$

and thus, using once again Eq.(9), the required result follows. ■

4 Deformed Covariant Derivative

Let $\langle U, \Gamma \rangle$ be a parallelism structure on U, and let ∇_a be the associated *a-DCDO*. By taking an invertible smooth extensor operator field λ on $V \supseteq U$, we can construct a deformed parallelism structure on U, denoted in [2] by $\left\langle U, \overset{\lambda}{\Gamma} \right\rangle$, with associated *a-DCDO* denoted by $\overset{\lambda}{\nabla}_a$.

As we know (see [2]), such deformed covariant derivative operator $\overset{\lambda}{\nabla}_a$has the following properties: for all $X \in \bigwedge \mathcal{V}(U)$, $\overset{\lambda}{\nabla}_a X = \underline{\lambda}(\nabla_a \underline{\lambda}^{-1}(X))$, and also for all $\Phi \in \bigwedge \mathcal{V}^\star(U)$, $\overset{\lambda}{\nabla}_a \Phi = \underline{\lambda}^{-\triangle}(\nabla_a \underline{\lambda}^{\triangle}(\Phi))$.

We present now two properties for $\overset{\lambda}{\nabla}_a$which are generalizations from the properties just recalled above.

- For all $\tau \in ext_k^l \mathcal{V}(U)$:

$$\overset{\lambda}{\nabla}_a \tau = \underline{\lambda} \nabla_a \underline{\lambda}^{-1} \tau, \tag{22}$$

where $\underline{\lambda}^{-1}\tau$ means the action of $\underline{\lambda}^{-1}$ on the smooth multivector extensor field τ, and $\underline{\lambda}\nabla_a\underline{\lambda}^{-1}\tau$ is the action of $\underline{\lambda}$ on the smooth multivector extensor field $\nabla_a\underline{\lambda}^{-1}\tau$.

- For all $\upsilon \in \overset{*}{ext}{}_k^l \mathcal{V}(U)$:

$$\overset{\lambda}{\nabla}_a \upsilon = \underline{\lambda}^{-\triangle} \nabla_a \underline{\lambda}^{\triangle} \upsilon, \tag{23}$$

where $\underline{\lambda}^{\triangle}\upsilon$ means the action of $\underline{\lambda}^{\triangle}$ on the smooth multiform extensor field υ, and $\underline{\lambda}^{-\triangle}\nabla_a\underline{\lambda}^{\triangle}\upsilon$ is the action of $\underline{\lambda}^{-\triangle}$ on the smooth multiform extensor field $\nabla_a\underline{\lambda}^{\triangle}\upsilon$.

Proof. We will prove only the property for smooth multivector extensor fields as given by Eq.(22). Without restrictions on generality, we can work with a multivector extensor field $(X, \Phi) \longmapsto \tau(X, \Phi)$.

Then, using Eq.(9) and taking into account the properties just recalled above, we can write

$$
\begin{aligned}
&(\overset{\lambda}{\nabla}_a\tau)(X,\Phi)\\
&=\overset{\lambda}{\nabla}_a\tau(X,\Phi)-\tau(\overset{\lambda}{\nabla}_aX,\Phi)-\tau(X,\overset{\lambda}{\nabla}_a\Phi)\\
&=\underline{\lambda}(\nabla_a\underline{\lambda}^{-1}\circ\tau(X,\Phi))-\tau(\underline{\lambda}(\nabla_a\underline{\lambda}^{-1}(X)),\Phi)-\tau(X,\underline{\lambda}^{-\triangle}(\nabla_a\underline{\lambda}^{\triangle}(\Phi))),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&(\underline{\lambda}^{-1}\circ\overset{\lambda}{\nabla}_a\tau)(X,\Phi)\\
&=\nabla_a\underline{\lambda}^{-1}\circ\tau(X,\Phi)-\underline{\lambda}^{-1}\circ\tau(\underline{\lambda}(\nabla_a\underline{\lambda}^{-1}(X)),\Phi)\\
&-\underline{\lambda}^{-1}\circ\tau(X,\underline{\lambda}^{-\triangle}(\nabla_a\underline{\lambda}^{\triangle}(\Phi))).
\end{aligned}
$$

By recalling the action of an extended operator $\underline{\lambda}^{-1}$ on a multivector extensor τ, we get

$$
\begin{aligned}
&(\underline{\lambda}^{-1}\circ\overset{\lambda}{\nabla}_a\tau)(X,\Phi)\\
&=\nabla_a\underline{\lambda}^{-1}\tau(\underline{\lambda}^{-1}(X),\underline{\lambda}^{\triangle}(\Phi))-\underline{\lambda}^{-1}\tau(\nabla_a\underline{\lambda}^{-1}(X)),\underline{\lambda}^{\triangle}(\Phi))\\
&-\underline{\lambda}^{-1}\tau(\underline{\lambda}^{-1}(X)),\nabla_a\underline{\lambda}^{\triangle}(\Phi)).
\end{aligned}
$$

Using once again Eq.(9), we have

$$
(\overset{\lambda}{\nabla}_a\tau)(X,\Phi)=\underline{\lambda}\circ(\nabla_a\underline{\lambda}^{-1}\tau)(\underline{\lambda}^{-1}(X),\underline{\lambda}^{\triangle}(\Phi)),
$$

and finally recalling once again the action of an extended operator $\underline{\lambda}$ on a multivector extensor $\nabla_a\underline{\lambda}^{-1}\tau$, the required result follows. ■

5 Relative Covariant Derivative

Let $\langle U_0,\Gamma\rangle$ be a parallelism structure on U_0, and let ∇_a be as usually its associated *a-DCDO*. Take any relative parallelism structure $\langle U,B\rangle$ compatible with $\langle U_0,\Gamma\rangle$ ($U_0\cap U\neq\emptyset$). Denote by ∂_a its associated *a-DCDO*. As we know (see [2]), there exists a well-defined smooth vector operator field on $U_0\cap U$, called the *relative connection field* γ_a, which satisfies the

split theorem valid for smooth multivector fields and for smooth multiform fields, i.e.: for all $X \in \bigwedge\mathcal{V}(U_0 \cap U)$,$\nabla_a X = \partial_a X + \underset{\smile}{\gamma_a}(X)$, and for all $\Phi \in \bigwedge\mathcal{V}^\star(U_0 \cap U)$, $\nabla_a \Phi = \partial_a \Phi - \underset{\smile}{\gamma_a^{\triangle}}(\Phi)$.

We now present a split theorem for smooth multivector extensor fields and for smooth multiform extensor fields, which are the generalizations of the properties just recalled above.

Theorem. (i) For all $\tau \in \overset{l}{\underset{k}{ext}}\,\mathcal{V}(U_0 \cap U)$:

$$\nabla_a \tau = \partial_a \tau + \underset{\smile}{\gamma_a}\tau, \tag{24}$$

where $\underset{\smile}{\gamma_a}\tau$ means the action of $\underset{\smile}{\gamma_a}$ on the smooth multivector extensor field τ.

(ii) For all $\upsilon \in \overset{*}{\overset{l}{\underset{k}{ext}}}\,\mathcal{V}(U_0 \cap U)$:

$$\nabla_a \upsilon = \partial_a \upsilon - \underset{\smile}{\gamma_a^{\triangle}}\upsilon, \tag{25}$$

where $\underset{\smile}{\gamma_a^{\triangle}}\upsilon$ means the action of $\underset{\smile}{\gamma_a^{\triangle}}$ on the smooth multiform extensor field υ.

Proof. We prove only the property for smooth multivector extensor fields, i.e., Eq.(24). Without loss of generality, we check this statement for a multivector extensor field $(X, \Phi) \longmapsto \tau(X, \Phi)$.

Using Eq.(9), and by taking into account the properties just recalled above, we have

$$\begin{aligned}
(\nabla_a \tau)(X, \Phi) &= \nabla_a \tau(X, \Phi) - \tau(\nabla_a X, \Phi) - \tau(X, \nabla_a \Phi) \\
&= \partial_a \tau(X, \Phi) + \underset{\smile}{\gamma_a}(\tau(X, \Phi)) \\
&- \tau(\partial_a X + \underset{\smile}{\gamma_a}(X), \Phi) - \tau(X, \partial_a \Phi - \underset{\smile}{\gamma_a^{\triangle}}(\Phi)) \\
&= \partial_a \tau(X, \Phi) - \tau(\partial_a X, \Phi) - \tau(X, \partial_a \Phi) \\
&+ \underset{\smile}{\gamma_a}(\tau(X, \Phi)) - \tau(\underset{\smile}{\gamma_a}(X), \Phi) + \tau(X, \underset{\smile}{\gamma_a^{\triangle}}(\Phi)).
\end{aligned}$$

Then, using once again Eq.(9) and recalling the action of a generalized operator $\underset{\smile}{\gamma_a}$ on a multivector extensor τ, see [1], we get

$$(\nabla_a \tau)(X, \Phi) = (\partial_a \tau)(X, \Phi) + (\underset{\smile}{\gamma_a}\tau)(X, \Phi),$$

and the proposition is proved. ■

6 Conclusions

Using the algebraic and analytical tools developed in previous papers [1, 2, 3] we presented a theory of covariant derivatives, deformed covariant derivatives and relative covariant derivatives of extensor fields, which are essential tools in geometric theories of the gravitational field (and also in the Lagrangian formalism of general field theory [4]), as will be shown in forthcoming papers.

References

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